Consistent refinement of Bethe strings for spin and electron models and a new non-Bethe solution

2002 J. Phys. A: Math. Gen. 35 L125

(http://iopscience.iop.org/0305-4470/35/10/101)

J. Phys. A: Math. Gen. **35** (2002) L125–L132 PII: S0305-4470(02)27887-7

LETTER TO THE EDITOR

Consistent refinement of Bethe strings for spin and electron models and a new non-Bethe solution

Anjan Kundu

Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Calcutta 700 064, India

E-mail: anjan@tnp.saha.ernet.in

Received 10 August 2001, in final form 26 November 2001
Published 1 March 2002
Online at stacks.iop.org/JPhysA/35/L125

Abstract
Well known Bethe strings, such as those in spin chains, the δ-electron gas, and the Hubbard and t–J models, are shown to be imprecise, while their consistent refinement along with a new non-Bethe r-string is discovered. A connection with earlier results is established and the string hypothesis problem is discussed in the light of the present findings .

PACS number: 03.65.Ge, 75.10.Jm, 05.90+m, 11.55.Ds

A new direction in physics, namely the theory of exactly solvable quantum systems, opened up following the classic work of Bethe [1]. Though Bethe's objective was to solve exactly the eigenvalue problem of the Heisenberg spin chain only, this *XXX* spin-$\frac{1}{2}$ model proved to be a generic one, and almost the same Bethe ansatz or its nested extensions to include particles with more degrees of freedom, e.g. electrons, were found to be applicable to a number of important models in condensed matter physics as well as in quantum field theory (see [2]). The celebrated Bethe ansatz assumes a specific form for the eigenvectors having parametric dependence on rapidity variables, which in turn are to be determined from specified equations known as the Bethe ansatz equations (BAE). Bethe also proposed a special type of complex solutions for BAE at the thermodynamic limit in the form of a string, known as the Bethe string. He showed further that such strings form a complete set by proving that their total number, together with the real roots, give exactly the required number of Bethe ansatz states. These Bethe string solutions have played an important role in all the Bethe ansatz solvable fundamental integrable models such as spin chains [3,4], the δ-function electron gas [5,6], the Hubbard model [7,8], the supersymmetric t–J model [9, 10], etc. The string solutions correspond to the bound states and describe excited states of the models. A popular conjecture, known as the string hypothesis, assumes that all complex solutions of BAE at the thermodynamic limit must be of the Bethe string type. although this conjecture has never really been proved, it has been applied successfully to all the above models for calculating important physical quantities like energy, entropy, pressure, magnetization, etc, through the thermodynamic Bethe ansatz (TBA).

However, in the early eighties some criticisms started appearing in this success story of the Bethe string when, in a series of studies, the string hypothesis apparently failed and, instead of the Bethe strings, non-Bethe complex solutions were reported to appear as close-roots in the form of quartets and 2-strings and as wide-roots with no restriction on their forms [11–14]. In a subsequent paper [15], an upper bound $\mathrm{Re}(\lambda_l) \ll \sqrt{N}$ on Bethe string roots was identified, beyond which significant deformation of strings was reported [16].

Despite these major criticisms of the string hypothesis, strangely enough no significant attention was given in subsequent years towards understanding or resolving this problem, and, as far as we know, the consistency of the Bethe string form itself has never been questioned before. Therefore, it is even more surprising for us to find that the well known Bethe string for all the fundamental models mentioned above, widely quoted and used for so many years, is in fact not a precise solution to the BAE even within its validity range and for large N. Having noted this, we are able to obtain here the precise and consistent form of the Bethe string by refining its correction terms, which therefore should replace the accepted string forms presented in a series of celebrated works [1, 3–10] dealing with the fundamental spin and electron models cited above. Moreover, we have discovered a new non-Bethe string with arbitrary r roots, compatible with the earlier observations. Based on the explicit forms of two kinds of strings, one could compare therefore their singularity structures appearing in the BAE, the energies of the states formed by them and hopefully all other thermodynamic quantities, which should contribute towards the solution of the longstanding problem of the string hypothesis.

Let us start by checking first the known Bethe string as solutions to the respective BAE at the thermodynamic limit. In all the spin and electron models mentioned above, the same BAE are obtained by both the coordinate and the algebraic Bethe ansätze, though in the former they are generated as a consequence of the periodic boundary condition on the wavefunctions [1], while in the later as the analyticity condition for eigenvalues of the transfer matrix [17]. The *XXX* spin-$\frac{1}{2}$ chain being a generic model, we intend to examine its solutions in more detail and draw conclusions about the other models by analogy.

XXX spin chain. The BAE for this model may be given as

$$V_{1/2}^{N}(\lambda_l) = \prod_{n \neq l} V_1(\lambda_l - \lambda_n) \qquad l = 1, \ldots, r \quad V_\alpha(\lambda) \equiv \frac{\lambda + \mathrm{i}\alpha}{\lambda - \mathrm{i}\alpha}. \tag{1}$$

The well known Bethe string (BS) solution to the above BAE, valid for large N, is usually given by

$$\lambda_l^{(\mathrm{BS})} = \lambda_0 + \mathrm{i}\eta \left(l - \frac{r+1}{2} \right) + \mathrm{i}O(\mathrm{e}^{-\alpha N}) \qquad l = 1, \ldots, r \quad \alpha > 0 \tag{2}$$

with $\eta = 1$. This form of solution suggests that exponentially small corrections $O(\mathrm{e}^{-\alpha N})$ are needed for large values of N, which presumably are of the same order due to the same α parameter appearing for all the roots. At $N \to \infty$ therefore, all corrections in (2) must vanish making it an apparently exact solution [17, 18]. Note that conjugate roots are included in (2) with $\lambda_l^* = \lambda_{r+1-l}$, $l = 1, \ldots, s$, where $s = \frac{r}{2}$ for r even and $s = \frac{r-1}{2}$ for r odd.

To follow the idea of this solution, we note that any complex solution with $\mathrm{Im}(\lambda) < 0$ always makes $|V_{1/2}(\lambda)| < 1$, which can be proved for the Bethe string with $l \leqslant s$ by inserting the explicit form (2) with arbitrary r:

$$\left| V_{1/2}\big(\lambda_l^{(\mathrm{BS})}\big) \right| = \left| \left(\frac{\lambda_0 + \mathrm{i}(l - \frac{r}{2})}{\lambda_0 + \mathrm{i}(l - \frac{r+2}{2})} \right) \right| = (1 + \kappa_l)^{-1/2} \equiv \mathrm{e}^{-v_l} < 1 \tag{3}$$

due to

$$\kappa_l = \frac{r+1-2l}{\lambda_0^2 + (\frac{r}{2}-l)^2} > 0.$$

Consequently, for large values of N the LHS of the BAE (1), i.e. $V_{1/2}^N(\lambda_l^{(\mathrm{BS})}) \sim O(\mathrm{e}^{-v_l N})$, must always vanish exponentially. Therefore for the Bethe string to be a consistent solution, the RHS must also contain an exponentially vanishing factor of the same order of smallness. By direct check one sees easily that in the simplest cases of $r = 2, 3$ this condition is fulfilled and hence, the Bethe string satisfies (1) for large N, as shown explicitly in [17–19].

However, though at this point it might seem natural to assume that the same argument must apply for any arbitrary r [17, 18], we will see by direct insertion that this analogy fails here and the RHS as such becomes inconsistent for the string form (2) starting from $r = 4$. One finds that among the factors in the RHS, one with the next higher root gives for the Bethe string (2)

$$V_1\big(\lambda_l^{(\mathrm{BS})} - \lambda_{l+1}^{(\mathrm{BS})}\big) = V_1\big(-\mathrm{i} + O(\mathrm{e}^{-\alpha N})\big) \sim O(\mathrm{e}^{-\alpha N})$$

an exponentially small term. One can immediately notice, however, that, for $l > 1$ and $r \geqslant 4$, another singular factor appears from the adjacent lower root yielding an exponentially large term having the same order:

$$V_1\big(\lambda_l^{(\mathrm{BS})} - \lambda_{l-1}^{(\mathrm{BS})}\big) = V_1\big(+\mathrm{i} + O(\mathrm{e}^{-\alpha N})\big) \sim O(\mathrm{e}^{\alpha N})$$

while the rest of the factors coming from other roots give only finite contributions. Multiplying all these factors we finally get the RHS of (1) as $O(\mathrm{e}^{-\alpha N})O(\mathrm{e}^{\alpha N}) \sim O(1)$ which, however, has a finite limit contradicting the vanishing LHS. Therefore we see that for all roots $\lambda_l^{(\mathrm{BS})}$, with $l = 2, \ldots, r-1$ and $r \geqslant 4$, having two adjacent neighbours $\lambda_{l\pm1}^{(\mathrm{BS})}$, the well known string form (2), as such, is not a consistent solution of the BAE (1), except only for the end-roots $l = 1, r$ and the real root, which cover also the cases $r = 2, 3$. For example, for $r = 4$, as worked out in our preliminary report [19], the above string form holds for the end-roots with $l = 1, 4$, but not for the roots with $l = 2, 3$.

XXZ spin chain. The BAE take the same form as in the *XXX* spin chain: $\tilde{V}_{\eta/2}^N(\lambda_l) = \prod_{n\neq l} \tilde{V}_\eta(\lambda_l - \lambda_n)$ but through a redefined function $\tilde{V}_\alpha(\lambda) \equiv \frac{\sin(\lambda+\mathrm{i}\alpha)}{\sin(\lambda-\mathrm{i}\alpha)}$, and the well known string solution for $\lambda_l^{(\mathrm{BS})}$ can be given again by (2) [3]. Skipping the details, we mention only that since the functions $\sinh x$ and x behave similarly at small x, the above reasoning for the *XXX* string is paralleled in this case and one encounters a similar mismatch for its standard string form.

Repulsive δ-function electron gas. For this and all other electron models considered below an additional set of rapidity variables is needed. The BAE is therefore extended to include another set of equations

$$\prod_{j}^{N_e} V_{c/2}(\lambda_l - k_j) = \prod_{n\neq l} V_c(\lambda_l - \lambda_n)$$

though its structure is very similar to (1) and the string solution for λ_l is given in the same form (2) with $\eta = c > 0$ and $\{k_j\}$ real [5, 6]. Since here the k_j's are real, for complex $\lambda_l^{(\mathrm{BS})}$ with $l \leqslant s$, the LHS becomes a product of terms each being < 1. Therefore, for the string solutions at large N_e, one gets a vanishing LHS, while the RHS having exactly the same form as in (1) remains finite as argued above.

Hubbard model. The additional BAE exhibit a very similar structure to the above electron model giving

$$\prod_j^{N_e} V_{U/4}(\lambda_l - \sin k_j) = \prod_{n\neq l} V_{U/2}(\lambda_l - \lambda_n).$$

However, in this case along with the string solutions for $\{\lambda_l, \lambda'_n\}$ in the form (2) with $\eta = \frac{1}{2}U$, M' pairs of solutions from $\{k_j\}$ can also be of string type satisfying $k_n^{\pm} - \lambda'_n = \mp\frac{1}{4}\mathrm{i}U$, $n = 1, \ldots, M'$ [7]. Note that in spite of $k_n^{\pm}$ being complex, due to the presence of their conjugates also in the factors of the LHS, any complex string root for λ_l, $l \leqslant s$, will make the factors < 1. Therefore for $N_e \to \infty$ the LHS $\to 0$, while the RHS, due to its same form as in the above discussed cases, gives a finite contribution for $r \geqslant 4$.

Supersymmetric t–J model. For a particular (BBF) type of excitations [10], one set of BAE takes the form

$$V_1^N(\lambda_l) \prod_\beta V_1(\lambda_l - \gamma_\beta) = \prod_{n\neq l} V_2(\lambda_l - \lambda_n).$$

The string solutions for λ_l are given again by (2) with all γ_β's real [10]. Noting the first factor in the LHS to be the same as in the *XXX* case and $\{\gamma_\beta\}$ being real, we conclude as before that the LHS $\to 0$ for $N \to \infty$. The RHS, however, again being the same as in the *XXX* case, gives non-vanishing terms, starting from $r \geqslant 4$.

As we see from the above arguments, the mismatch of the well known Bethe string form (2) is due to its imprecise correction terms. We therefore propose the consistent and precise form of the Bethe string (PBS) as

$$\lambda_l^{(\mathrm{PBS})} = \lambda_0 + \mathrm{i}\left(l - \frac{r+1}{2}\right) + \mathrm{i}O(\mathrm{e}^{-\alpha_l N}) \qquad l = 1, \ldots, s \quad \text{and} \quad \lambda_{r+1-l} = \lambda_l^* \tag{4}$$

provided the exponential orders in its correction terms are fine tuned as a strictly growing sequence

$$0 < \alpha_1 < \cdots < \alpha_l < \alpha_{l+1} < \cdots < \alpha_s \qquad \text{with} \quad \alpha_l - \alpha_{l-1} = v_l = \tfrac{1}{2}\ln(1+\kappa_l) > 0 \tag{5}$$

with κ_l as defined in (3). Note that the term $O(\mathrm{e}^{-\alpha_l N})$ in (4) stands for the terms like $c_l \mathrm{e}^{-\alpha_l N}$ with l-dependent multiplicative constants c_l. The essential point in proving the validity of this refined Bethe string is that, unlike the known form (2), the adjacent roots now contribute zeros and poles of different orders of smallness in the RHS of the BAE, since

$$V_1\big(\lambda_l^{(\mathrm{PBS})} - \lambda_{l+1}^{(\mathrm{PBS})}\big) = O(\mathrm{e}^{-\alpha_l N}) - O(\mathrm{e}^{-\alpha_{l+1} N}) \approx O(\mathrm{e}^{-\alpha_l N})$$

while

$$V_1\big(\lambda_l^{(\mathrm{PBS})} - \lambda_{l-1}^{(\mathrm{PBS})}\big) = \big(O(\mathrm{e}^{-\alpha_l N}) - O(\mathrm{e}^{-\alpha_{l-1} N})\big)^{-1} \approx O(\mathrm{e}^{\alpha_{l-1} N})$$

using the strict inequality (5). Therefore the RHS becomes $O(\mathrm{e}^{-\alpha_l N})O(\mathrm{e}^{\alpha_{l-1} N}) \sim O(\mathrm{e}^{-v_l N})$, i.e. consistent with the LHS, which has the same vanishing limit as before. It is important to note that, contrary to its known form (2), the correction terms present in the proper Bethe string (4) are rather complicated with simultaneous involvement of small terms of all different orders and none of them can be neglected from the beginning, even at $N \to \infty$.

We emphasize again that, the *XXX* spin chain being a generic case, the same consistent Bethe string (4) with refinement (5) will be equally valid for all fundamental integrable models discussed above and should therefore replace the corresponding well known and widely used

Bethe strings appearing in related works cited above. Curiously, however, this seems not to affect string-based physical results obtained through the TBA. The TBA method [3, 5] apparently needs not the solution of an individual BAE, but the solution of a product of several BAEs with different roots for the same string:

$$\prod_{\alpha} V_{1/2}^{N}(\lambda_{\alpha}^{(1)(\mathrm{BS})}) = \prod_{\alpha,j} V_1(\lambda_{\alpha}^{(1)(\mathrm{BS})} - \lambda_{j}^{(2)(\mathrm{BS})}).$$

Note that since the complex conjugate of each root is also present in the product, its LHS remains finite even for large N and therefore the standard string solutions hold for such a product-BAE. Therefore, despite being of imprecise form for the individual BAE, the known Bethe string is fortunately capable of producing correct TBA results.

Nevertheless, the precise form of the Bethe string we find here is important not only as a consistent form of solution for the BAE but also for its possible comparison with a non-Bethe solution. For comparison one also needs an explicit non-Bethe string with arbitrary r roots, which, however, is not available in the literature. We therefore propose as well a new non-Bethe string (NBS) solution to the BAE (1) in the form

$$\lambda_l^{(\mathrm{NBS})} = \lambda_0 + \mathrm{i}\frac{1}{s}\left(l - \frac{r+1}{2}\right) + \mathrm{i}O(\mathrm{e}^{-\alpha N}) \qquad l = 1, \ldots, r \quad \alpha > 0 \tag{6}$$

with $s = \frac{1}{2}r$ for even r and $s = \frac{1}{2}(r-1)$ for odd r. Note that the correction terms in (6) are much simpler compared with the Bethe case and moreover they can be regulated by the background real roots, the presence of which is essential, as we see below, for the survival of the non-Bethe strings. For simplicity we consider only the case of even r; the odd r case can be handled in a similar way. It is easy to check that each root $\lambda_l^{(\mathrm{NBS})}$ with $l \leqslant s$ for string (6) yields as before the LHS $= O(\mathrm{e}^{-v_l N})$, now with $v_l = \frac{1}{2}\ln(1+\kappa_l)$, $\kappa_l = \frac{1}{D}(r+1-2l)$, $D = \frac{1}{2}r(\lambda_0^2 + \frac{1}{r^2}(2l-1-\frac{1}{2}r)^2)$, while the RHS having only one partner, $\tilde{\lambda}_l^{(\mathrm{NBS})} = \lambda_{l+s}^{(\mathrm{NBS})}$, contributes also with a vanishing term of the order $V_1(\lambda_l^{(\mathrm{NBS})} - \tilde{\lambda}_l^{(\mathrm{NBS})}) = O(\mathrm{e}^{-\alpha N})$. However, this apparent consistency breaks down if we consider the product of this equation with that of its partner, giving in the LHS again an exponentially vanishing term $|V_{1/2}(\lambda_l^{(\mathrm{NBS})})V_{1/2}(\tilde{\lambda}_l^{(\mathrm{NBS})})|^N \approx O(\mathrm{e}^{-(v_l - v_{l+s})N})$ for large N and $l < \frac{1}{4}(2+r) \equiv l^*$. For $l > l^*$ this term similarly blows up exponentially. However, one finds now that the singular terms in the corresponding RHS are mutually cancelled due to opposite but equal contributions from the lth root and its partner, i.e. the $(l+s)$th root, which therefore leads the RHS to a finite limit and thus to a contradiction. Therefore we conclude that, unlike the Bethe case, such non-Bethe strings cannot survive in isolation and must be coupled with background real roots, which can make ansatz (6) consistent by supplying the necessary vanishing (exploding) terms in the RHS. This also shows perhaps at the microscopic level why the non-Bethe strings were observed earlier only for the aniferromagnetic ground state (AFGS) [12]. To demonstrate this we couple (6) with M number of real roots $\{\nu_a\}$ and notice that, due to the appearance of additional terms like $\prod_a^M |V_1(\lambda_l^{(\mathrm{NBS})} - \nu_a)|$ in the RHS for the lth root of the string, a new exponentially diminishing (increasing) term may arise and since each of its constituent terms are either < 1 or > 1, by multiplying them one gets the contribution from M number of such terms as $O(\exp(-\sum_a^M \phi_{la}))$, where $\phi_{la} = \frac{1}{2}\ln(1+\kappa_{la})$, with $\kappa_{la} = \frac{1}{\bar{D}}(r+1-2l)$, $\bar{D} = \frac{r}{4}((\lambda_0 - \nu_a)^2 + \frac{1}{r^2}(2l-1)^2)$. Therefore, for large M there appears now a possibility to compensate for the singular term arising in the LHS of the product BAE mentioned above and thus to resolve the controversy for the existence of strings like (6). The exact values of ν_a should in fact be determined self-consistently from the BAE. However, this is difficult to achieve in practice for large values of M, N, even numerically and such strings can exist only for large N, M. Therefore we make here only some estimates to show the right trend. Since the product of the BAEs for the lth root with its partner leading to

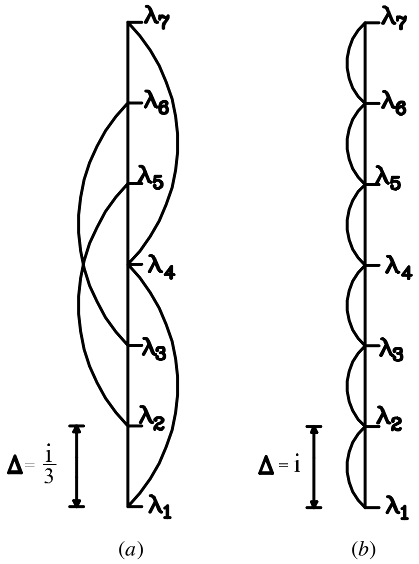

Figure 1. Arrangement of roots in string solutions with $r = 7$. (*a*) Non-Bethe string with length $L(7) = 6\Delta = 2\mathrm{i}$ represents a close-root form and breaks up into a quartet with $\{\lambda_2, \tilde{\lambda}_2(= \lambda_5), \lambda_2^*(= \lambda_6), \tilde{\lambda}_2^*(= \lambda_3)\}$ and a triplet with $\{\lambda_1, \tilde{\lambda}_1(= \lambda_4) = \tilde{\lambda}_1^*, \lambda_1^*(= \lambda_7)\}$. (*b*) Bethe string with length $L(7) = 6\Delta = 6\mathrm{i}$ in the wide-root region cannot break up into smaller units since each root has two partners, each of which in turn has a different partner.

a contradiction for the isolated NBS is our main concern, we observe first that each of the additional terms in its RHS, e.g. $\mathrm{e}^{-(\phi_{la}-\phi_{(l+s)a})}$ exhibits diminishing (increasing) trend also for $l < l^*(l > l^*)$, matching exactly with the behaviour of the LHS established above. This shows at least qualitatively that for the NBS the coupling with real roots can indeed yield necessary singular terms in the RHS that might match such terms appearing in the LHS. To obtain further quantitative estimates we have checked numerically the ratios of the singular terms arising in both the LHS and RHS and sought the real solutions of ν_a when they match. We find in particular that for various values of r and l there always exists a pair of real solutions for ν_a to the relation $\mathrm{e}^{-2(v_l-v_{l+s})} \approx \mathrm{e}^{-(\phi_{la}-\phi_{(l+s)a})}$ taking one positive and another negative value. For $\lambda_0 = 0$, for example, we get ν_a placed symmetrically around the origin, which gives the total momentum of the system: $P = 2\pi \times$ integer, in agreement with the translational invariance of the system. Therefore for $M = \frac{N}{2}$ and a distribution of ν_a's having the same order of magnitude, one is likely to match the singularities in both sides of the BAE allowing the NBS (6) to exist. Note that in the thermodynamic limit this distribution of real roots should correspond to the AFGS and the NBS should correspond to the earlier non-Bethe string results observed also over the AFGS [11–14].

Now we focus on some interesting properties of NBS (6) and show its further resemblance to earlier observations. Firstly, we notice that due to the reduced inter-root distance $\Delta = \frac{\mathrm{i}}{s}$ its length $L(r) = \mathrm{i}\frac{(r-1)}{s}$ is bounded as $1 \geqslant |L(r)| \geqslant 2$ and in contrast to the Bethe string exhibits therefore a close-root form, which is in accordance with the earlier observations. It is also evident that together with each root (λ_l) its partner ($\tilde{\lambda}_l$), its conjugate (λ_l^*) and the partner of its conjugate ($\tilde{\lambda}_l^*$) form a closed unit of four and being decisive contributors to the equation at large N, become almost independent entries at the thermodynamic limit. These groups of four may also get reduced to form a doublet or a triplet due to possible degeneracies (see

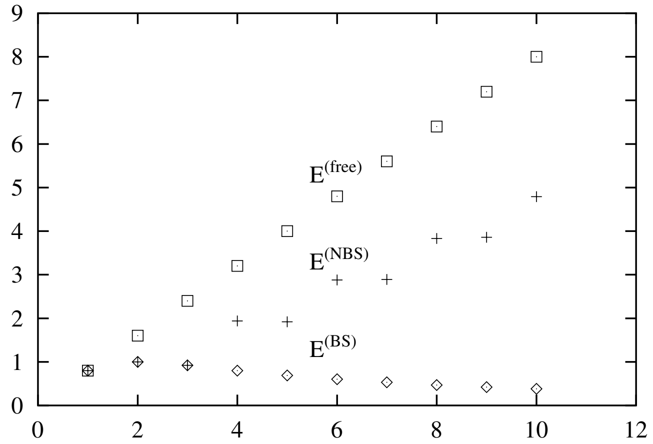

Figure 2. Comparison of energies for $r = 1, 2, \ldots, 10$: $E_r^{(\mathrm{BS})}$ corresponding to the r-Bethe string, $E_r^{(\mathrm{NBS})}$ to the r-non-Bethe string, and $E_r^{(\mathrm{free})}$ to the r free magnons, showing both strings as bound states with $E_r^{(\mathrm{free})} \geqslant E_r^{(\mathrm{NBS})} \geqslant E_r^{(\mathrm{BS})}$.

figure 1). However, this can occur only once, and that also when r is not divisible by 4. Thus the close-root string (6) may split up into units of four, three and two at the thermodynamic limit, reproducing again the non-Bethe structures observed earlier. Moreover, we see that for each of the roots with $l < l*$, which corresponds to $|\mathrm{Im}(\lambda_l)| > \frac{1}{2}$, its partner $\tilde{\lambda}_l$ is always with $|\mathrm{Im}(\tilde{\lambda}_l)| < \frac{1}{2}$. This fact also mimics amazingly the observation of [13] stating that the close roots of type I must have their partners from among the close roots of type II.

Due to such striking agreement with earlier studies, we hope that the more general non-Bethe string structure (6) should survive also in the thermodynamic limit and should be consistent with the integral equations derived earlier for the complex roots in the anti-ferromagnetic case. The precise and explicit forms of the general Bethe (PBS) (4) and non-Bethe (NBS) (6) r strings found here should also be useful for comparing their corresponding properties. As we have noticed, their correction terms induce a different nature of singularity structures in the BAE at large N. Generically roots of the Bethe string (PBS) (4) produce zeros and poles of different orders of smallness in the BAE, arranged in a growing sequence and the roots are located mostly in the wide-root region: $|\mathrm{Im}(\lambda_l)| > 1$. The non-Bethe string (NBS) (6) on the other hand, allows the singularities of its roots to be of the same order of smallness and the roots themselves are concentrated only in the close-root region. Therefore, if one looks for the complex roots of the BAE at the AFGS with their singularities having the same order in the whole complex plane, then among the string solutions only the close-root NBS are likely to appear, which supports the earlier findings. Similarly the energies of the states created by Bethe and non-Bethe r-strings can also be compared using their explicit forms giving, respectively,

$$E_r^{(\mathrm{BS})} = \frac{r}{\lambda_0^2 + (\frac{r}{2})^2} \qquad \text{and} \qquad E_r^{(\mathrm{NBS})} = \sum_{j=1}^{s} \frac{2g_j}{\lambda_0^2 + g_j^2}$$

where the factor $g_j = g_1 - \frac{1}{s}(j-1)$, with $g_1 = \frac{3}{2}$ for odd r and $g_1 = \frac{3}{2} - \frac{1}{2s}$ for even r. One can show analytically for $r \to \infty$ and numerically for finite r (see figure 2) that $E_r^{(\mathrm{BS})}$ is always lower than $E_r^{(\mathrm{NBS})}$, while both have lower values than that of the r-free magnons:

$$E_r^{(\mathrm{free})} = \frac{r}{\lambda_0^2 + \frac{1}{4}}.$$

Therefore one concludes that, though both Bethe and non-Bethe strings may give bound states, the non-Bethe ones must be more loosely bound with higher energies. At this stage, however, conclusive statements are still difficult to make regarding the string hypothesis problem. One perhaps should consider the TBA analysis using the general r-NBS over the AFGS, which we leave as a future problem.

I would like to thank Professor Deepak Dhar and Professor Andreas Klümper for stimulating discussions. I also acknowledge with thanks the hospitality of the Institute for Theoretical Physics, Dortmund University, where this work was completed and both financial and other support from the AVH Foundation, Germany.

References

[1] Bethe H 1931 *Z. Phys.* **71** 205
[2] Korepin V and Essler (ed) 1994 *Exactly solvable models of strongly correlated electrons* (Singapore: World Scientific)
Kulish P and Sklyanin E 1982 *Lecture Notes in Physics* **151** 61
[3] Takahashi M 1971 *Prog. Theor Phys.* **46** 401
[4] Gaudin M 1971 *Phys. Rev. Lett.* **26** 1301
[5] Takahashi M 1971 *Prog. Theor. Phys.* **46** 1388
[6] Lai C K 1971 *Phys. Rev. Lett.* **26** 1472
[7] Takahashi M 1972 *Prog. Theor. Phys.* **47** 69
[8] Essler F, Korepin V and Schoutens K 1992 *Nucl. Phys.* **384** 431
[9] Schlottmann P 1987 *Phys. Rev.* B **36** 5177
[10] Foerster A and Karowski M 1993 *Nucl. Phys.* B **396** 611
[11] Destri C and Lowenstein J H 1982 *Nucl. Phys.* B **205** 71
[12] Woynarovich F 1982 *J. Phys. A: Math. Gen.* **15** 2985
[13] Babelon O, de Vega H and Viallet C 1982 *Nucl. Phys.* B **220 [FS8]** 13
[14] Woynarovich F 1983 *J. Phys. C: Solid State Phys.* **16** 5293
[15] Vladimirov A A 1984 *Phys. Lett.* A **105** 418
[16] Dhar A and Shastry B S 2000 *Phys. Rev. Lett.* **85** 2813
[17] Takhtajan L A and Faddeev L D 1981 *Zap. Nauch. Sem. LOMI* **109** 134
[18] Izumov Yu A and Skryabin Yu N 1988 *Statistical Mechanics of Magnetically Ordered Systems* (New York: Consultant Bureau) p 197
[19] Kundu A 2000 Refinement of Bethe ansatz string solution and its alternative *Preprint* arXiv cond-mat/0012375